Summary of Lecture 15

Modes of Lateral Flight Dynamics and Their Approximations: Dutch Roll, Spiral, and Rolling Modes

1. Recall the state space model describing lateral flight dynamics

\[
\begin{align*}
\begin{bmatrix}
\dot{\psi} \\
\dot{\phi}^T \\
\dot{r} \\
\dot{p}
\end{bmatrix} &=
\begin{bmatrix}
Y_v & Y_p & Y_r - u_0 & g & 0 \\
L_v' & L_p' & L_r' & 0 & 0 \\
N_v' & N_p' & N_r' & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\psi \\
\phi \\
r \\
p
\end{bmatrix}

+ 
\begin{bmatrix}
Y_{\delta_s} & Y_{\delta_r} \\
L_{\delta_s} & L_{\delta_r} \\
N_{\delta_s} & N_{\delta_r} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_s \\
\delta_r
\end{bmatrix}
\end{align*}
\]

where

\[
L' = \frac{L + \frac{b}{T_z} N}{1 - \frac{T_z}{T_z}} \quad \text{and} \quad N' = \frac{N + \frac{b}{T_z} L'}{1 - \frac{T_z}{T_z}}
\]

Note that if \( I_{xz} = 0 \), then \( L' = L \) and \( N' = N \). Also, note that \( \beta = \frac{v}{u_0} \). In addition, the last equation \( \dot{\psi} = \dot{r} \) is decoupled and the system can further be reduced to a 4th order one.

2. Consider the following values for the lateral stability derivatives of the general aviation airplane as given in Example Problem 5.3, pp. 198-201 in [1]

\[
Y_v = -0.254, \quad Y_p = 0, \quad Y_r = 0, \quad L_v = -0.091, \quad L_p = -8.4, \quad L_r = 2.19, \quad N_v = 0.025, \quad N_p = -0.35 \quad \text{and} \quad N_r = -0.76.
\]

Also,

\[
u_0 = 176 \text{ ft/s}, \quad b = 33.4 \text{ ft}, \quad \text{and} \quad I_{xz} = 0
\]

Thus, we can construct the \( A \) matrix and, hence, determine its eigenvalues and eigenvectors

\[
\lambda_{DR} = -0.49 \pm 2.31i, \quad \lambda_s = -0.0098, \quad \lambda_r = -8.43
\]

Three typical lateral modes are identified; one oscillatory mode and two aperiodic modes: (i) Dutch Roll Mode \( (\omega_{n_{DR}} = 2.37 \text{ rad/s}, \zeta_{DR} = 0.21) \), (ii) Spiral Mode, and (iii) Pure Rolling Mode.

3. Eigenvector Analysis and Approximations: The corresponding normalized eigenvectors are

\[
\begin{align*}
\begin{bmatrix}
\Delta \beta \\
\Delta \phi \\
\Delta p_{b/2u_0} \\
\Delta r_{b/2u_0}
\end{bmatrix}_{DR} &=
\begin{bmatrix}
0.2319 \pm 1.1846i \\
-0.0461 \pm 0.2196i \\
-0.2377 \pm 0.0770i \\
1.0
\end{bmatrix}, \\
\begin{bmatrix}
\Delta \beta \\
\Delta \phi \\
\Delta p_{b/2u_0} \\
\Delta r_{b/2u_0}
\end{bmatrix}_{s} &=
\begin{bmatrix}
0.0292 \\
-0.0099 \\
0.0167 \\
1.0
\end{bmatrix}
\end{align*}
\]

and

\[
\begin{align*}
\begin{bmatrix}
\Delta \beta \\
\Delta \phi \\
\Delta p_{b/2u_0} \\
\Delta r_{b/2u_0}
\end{bmatrix}_r &=
\begin{bmatrix}
-0.0648 \\
-0.8002 \\
-0.0330 \\
1.0
\end{bmatrix}
\end{align*}
\]
- Pure Rolling Mode: The roll mode is the simplest and the most clear one among the three modes. It consists of almost pure roll with quite fast time constant ($\tau = 1/\lambda$ and $T_s = 4\tau$). So, we determine a very good approximation for the rolling mode as

$$\dot{p} = \mathcal{L}_p^r p + \mathcal{L}_\delta \dot{\delta}$$

Thus, the transfer function $\frac{p}{\delta}$ is given by $\frac{p}{\delta} = \frac{\mathcal{L}_p^r}{s - \mathcal{L}}$. Hence, the rolling mode eigenvalue is approximated as $\lambda_r \simeq \mathcal{L}_p^r$, which results in $\lambda_r \simeq -8.4$ for the general aviation airplane example; that is, is very close to the exact eigenvalue 8.43 (less than 1% error).

- Dutch Roll Mode: The Dutch roll mode incorporates sideslip, roll and yaw, with sideslip being the most significant. The common, though not so accurate, way of finding a Dutch roll approximation is to include $r$ with $\beta$. That is, we write

$$(\dot{v} \dot{r}) = \begin{bmatrix} Y_v & Y_r - u_0 \\ -N'_v & -N'_r \end{bmatrix} (v \ r) + \begin{bmatrix} Y_{\delta_v} & Y_{\delta_r} \\ N'_{\delta_v} & N'_{\delta_r} \end{bmatrix} (\delta_v \ \delta_r)$$

Its characteristic equation is given by

$$\det(\lambda I - A_{DR}) = \lambda^2 - (Y_v + N'_v)\lambda + Y_vN'_v - Y_rN'_r + u_0N'_v$$

That is,

$$\omega_{n_{DR}} = \sqrt{Y_vN'_v - Y_rN'_r + u_0N'_v} = \frac{Y_{\beta}N'_\beta - Y_rN'_\beta}{u_0} + N'_\beta$$

and $\zeta_{DR} = -\frac{Y_v + N'_v}{2\omega_{n_{DR}}} = -\frac{Y_{\beta} + u_0N'_v}{2u_0\omega_{n_{DR}}}$

This approximation results in the following characteristics

$$\lambda_{DR} = -0.51 \pm 2.08i \Rightarrow \omega_{n_{DR}} = 2.14rad/s, \ \zeta_{DR} = 0.24$$

for the general aviation airplane example. The natural frequency and damping ratio are off by about 9% and 12%, respectively.

Dutch Roll mode is objectionable to pilots and passengers and usually needs artificial damping through feedback control (stability augmentation). This artificial damping system is usually called Yaw Damper.

- Spiral Mode: It is not an easy to think of a single-state approximation to the spiral mode. However, if we rewrite its eigenvector to include the $\Delta \psi$-value, it will be

$$\begin{pmatrix} \Delta \beta \\ \Delta \beta / 2u_0 \\ \Delta \phi / 2u_0 \\ \Delta \phi / 2u_0 \\ \Delta \phi / 2u_0 \\ \Delta \psi \end{pmatrix} = \begin{pmatrix} 0.0292 \\ -0.0009 \\ 0.0167 \\ 1.0 \\ -17.8759 \end{pmatrix}$$

Therefore, if we compare the changes in angles $\beta$, $\phi$, and $\psi$, we find that the mode is mainly associated with yawing and rolling with a negligible side slip. As for the aerodynamic variables $\beta$, $pb/2u_0$, and $rb/2u_0$, they are almost negligible. That is, the aerodynamic forces in this
mode are very small, which is consistent with its long time constant. From this discussion, the candidate equation that can provide an approximation for the spiral mode is the yawing moment equation (\(\dot{r}\)-equation). In addition, \(p\) experiences the least change in the spiral mode, it is appropriate to neglect the \(p\)-changes. Then, the common spiral mode is obtained by ignoring the first equation and combining the second and third equations to give

\[
\dot{r} = \left( N'_r - \frac{C'_r}{C'_v} N'_v \right) r = \left( \frac{N'_r L'_\beta - L'_r N'_\beta}{C'_\beta} \right) r \quad \approx \lambda_s
\]

which results in \(\lambda_s \approx -0.16\) for the general aviation airplane example. Although the spiral approximation is inaccurate, it gives an idea about the relevant stability derivatives.

4. Description of Spiral Mode Instability: If the airplane experiences a positive sideslip disturbance (drift to the right), the static stability requirements are \(C_{N_\beta} > 0\) and \(C_{L_\beta} < 0\). That is, a good airplane should (a) yaw to the right to be aligned with its velocity vector again, and (b) roll to the left because negative rolling eventually leads to a decrease in \(\beta\) by virtue of weight (See the fourth term in the \(Y\)-force equation).

Spiral instability (divergence) occurs when the roll/lateral stability is not adequate (\(C_{L_\beta}\) is not negative enough), while directional stability is strong (\(C_{N_\beta}\) is a relatively-large positive number). As such, the airplane will have a relatively-strong yaw to the right satisfying (a). Meanwhile, the airplane will experience two rolling mechanisms; a weak one to the left due to the weak \(C_{L_\beta}\) and a stronger one to the right due to the induced roll due to yaw via \(C_{L_\beta} > 0\). As such, the airplane will roll to the right dissatisfying (b). The net result will be a right turning maneuver with zero sideslip. The turning maneuver gets sharper with time and usually results in a high-speed spiral dive if not corrected. Figure 1 shows the first ten seconds of the simulation due to a \(5^\circ\) sideslip disturbance for the general aviation airplane when its lateral stability is decreased 10 times.

5. Spiral Mode and Airplane Configuration: We can use the spiral mode approximation to write the following criterion for spiral stability

\[
N'_r L'_\beta - L'_r N'_\beta > 0
\]

That is, we may require \(C_{L_\beta} C_{N_\rho} > C_{L_\beta} C_{N_\rho}\). Note that increasing the vertical tail size will lead to an increase in both of area \(C_{N_\rho}\) and \(C_{N_\rho}\) together and, hence, may not affect spiral stability. It is mainly the lateral/roll stability coefficient \(C_{L_\beta}\) that can manipulate spiral stability. However, it is found that increasing lateral stability (more negative \(C_{L_\beta}\)) leads to a more unstable Dutch Roll mode, which is more annoying and dangerous than the slow spiral mode. For example, if the roll stability coefficient for the general aviation airplane example is augmented 20 times, the following eigenvalues are obtained

\[
\lambda_{DR} = 0.43 \pm 4.52i, \quad \lambda_s = -0.21, \quad \text{and} \quad \lambda_r = -10.06
\]

Figure 2 shows the response due to a \(5^\circ\) sideslip disturbance in that case.

The coupling term \(C_{L_\rho}\) is proportional to the lift coefficient of the wing. Hence, the above inequality may be satisfied for low-angle-of-attack, high-speed flight condition but may not be satisfied at a high-angle-of-attack, low-speed flight condition leading to spiral instability. Even
Figure 1: Demonstration of Spiral Divergence.

Figure 2: Dutch Roll Instability due to very large lateral/roll stability $C_{L\beta}$. 
though, too much lateral stability is undesirable in airplane design, as pilots can tolerate spiral instability well because of its slow time scale, while Dutch Roll instability may be dangerous. Increasing directional stability $C_{N_d}$ leads to a more unstable spiral mode. Increasing $C_{N_e}$ leads to a better damped Dutch Roll mode. However, it cannot be achieved without increasing $C_{N_d}$. So, artificial damping (Yaw Damper) is usually employed to achieve such a target.

6. **Reading for next lecture**: Sec. 4.7, and Sec. 5.5 in Nelson [1].

**References**