Controllability of Dynamical Systems: Overview

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My Education and Research

Education:

- Bachelor degree in electrical engineering, Alexandria University, Egypt.
- Post graduate studies in engineering mathematics, Alexandria University, Egypt.
- Ph.D. in applied and industrial mathematics, New Mexico Tech, USA.

Research: focused on applied, engineering, and industrial mathematics.

- Research areas: mathematical modeling, dynamical systems, differential equations, control theory, numerical methods and simulations, signal processing, and mathematical imaging.
- Computational tools: Matlab, Simulink, and Maple.
Outline of Today’s Talk

1. Tools and topics involved in controllability of systems: Dynamical Systems + Differential Geometry + Control Theory
   - Dynamical Systems
   - Differential Geometry
   - Control Theory

2. Controllability of systems
   - Controllability of linear systems
   - Controllability of nonlinear systems
   - Small Time Local Controllability

3. Research on Nonlinear Controllability: Theory and Applications
   - Theory and Applications
Tools and topics involved in controllability of systems: Dynamical Systems + Differential Geometry + Control Theory

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Research on Nonlinear Controllability: Theory and Applications
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Dynamical Systems are either Ordinary/Partial Differential Equations (DEs) that model or exactly describe changes of systems with respect to "Time".

Mathematical analysis for DEs is mostly topics on Stability, Boundedness, Existence, Uniqueness, and Numerical approximations.

ODE theory fundamentals are more well-established compared to PDE theory fundamentals. For example:
- Existence and Uniqueness conditions are more generalized for ODEs.
- Numerical analysis is far ahead in ODEs vs. PDEs.
- Systems of ODEs can be easily solved numerically vs. systems of PDEs.
Let us consider the nonlinear system:

\[
\frac{dX}{dt} = f(X, t), \quad X(t_0) = X_0
\]  

1. If \( f \) is continuous function of \( X \) and \( t \), then existence of solution is guaranteed.

2. Lipschitz continuity \( \|f(X_1, t) - f(X_2, t)\| \leq L \|X_1 - X_2\| \) is essential for uniqueness (e.g. Picard Lindelf theorem).

3. Stability of steady states/equilibriums can be studied by eigenvalue/linearization analysis and Lyapunov function.

Lyapunov function tells about stability if there exists a scaler function \( V : \mathbb{R}^n \to \mathbb{R} \) such that \( \nabla V \cdot f \) is locally negative or positive definite.
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Basics and Tools: Manifolds

- Manifold is a topological space that locally resembles Euclidean space near each point. More precisely, each point of an n-dimensional manifold has a neighbourhood that is homeomorphic to the Euclidean space of dimension n.
- A manifold can be constructed by giving a collection of coordinate charts, that is a covering by open sets with homeomorphisms to a Euclidean space.
- The example of the circle:

\[
X_{top}(x, y) = x \\
X_{bottom}(x, y) = x \\
X_{right}(x, y) = y \\
X_{left}(x, y) = y
\]
Basics and Tools: Tangent Spaces

- **Tangent space** $T_xM$: The space that contains all possible tangent vectors to a point $x$ in $n$-manifold $M$.
- Geometrically, the tangent space $T_xM$ can be seen as a linear approximation of $M$ at and around $x$.
- The tangent bundle $TM$: The union of all possible tangent spaces for all points of $M$:

$$TM = \bigcup_{x \in M} T_xM$$
Lie Derivative and Bracket

- Lie derivative is a formula for how a vector field $g$ changes on the flow of another vector field $f$.
- Lie derivative of $g$ with respect to $f$ is given by:

$$L_f g = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = [f, g]$$

(2)
Lie Derivative/Bracket Properties

- Lie derivative/bracket is a vector.
- Lie brackets have the following properties:
  - \([f, g] = -[g, f]\).
  - \([[[f + g], K] = [f, K] + [g, K]\).
  - \([[f, g], K] + [[f, k], g] + [[g, k], f] = 0\) (Jacobi-Identity).
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Let us consider the control-affine system:

\[
\frac{dX}{dt} = f(X(t)) + \sum_{i=1}^{m} g_i(X(t))u_i
\]  

(3)

where the state space \( X : I \subset \mathbb{R} \rightarrow M \), \( u \in U \subset \mathbb{R}^m \), and \( f, g_1...g_m \) are vector fields.

Let \( \Sigma = (M, F, U) \) with \( F = \{f, g_1...g_m\} \) represent a control-affine system. For a point \( x_0 \in M \), we define:

- \( R_\Sigma(x_0, T) \) for some \( T > 0 \) denotes the set of points reachable from \( x_0 \) by controlled trajectories of \( \Sigma \) in an exact time of \( T \).
- \( R_\Sigma(x_0, \leq T) \) for some \( T > 0 \) denotes the set of points reachable from \( x_0 \) in at most \( T \) \( (R_\Sigma(x_0, \leq T) = \bigcup_{t \in [0, T]} R_\Sigma(x_0, t)) \).
- \( R_\Sigma(x_0) \) denotes all reachable points from \( x_0 \).
Definitions

The control-affine system $\Sigma$ is said to be

- **Accessible** from $x_0$ if there exists some $T > 0$ such that $\text{int}(R_\Sigma(x_0, \leq T)) \neq \phi$.

- **Controllable** from $x_0$ if for each $x \in M$ there exists some $T > 0$ such that $x \in R_\Sigma(x_0, T)$.

- **Small Time Local Controllable (STLC)** from $x_0$ if there exists some $T > 0$ such that $x_0 \in \text{int}(R_\Sigma(x_0, \leq t)) \neq \phi$, where $t \in [0, T]$. 
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Kalman Criterion

Let us consider the linear control-affine system:

\[
\frac{dX}{dt} = AX + Bu, \quad X \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad \text{(input vector)} \quad (4)
\]

where \( A \) is \( n \times n \) matrix and \( B \) is \( n \times m \).

**Theorem**: A necessary and sufficient condition for the system (4) to be controllable is to have:

\[
\text{rank}(\delta) = \text{rank}([B|BA|BA^2|...|BA^{n-1}]_{n\times nm}) = n
\]

**Proof**: By solving the system (4) and evaluate at \( T \) we get:

\[
X(T) = e^{AT}x_0 + \int_0^T e^{A(T-t)}Bu(t)dt \quad (5)
\]

Then we can rearrange it to be:

\[
e^{-AT}X(T) - x_0 = \int_0^T e^{-At}Bu(t)dt \quad (6)
\]
Using the Cayley Hamilton theorem we have:

\[ e^{-At} = \sum_{i=0}^{n-1} \alpha_i(t)A^i \]  \hspace{1cm} (7)

Then we can rewrite Eq (9) to be:

\[
\begin{bmatrix}
\int_0^T \alpha_0 u(t) \, dt \\
\cdot \\
\cdot \\
\int_0^T \alpha_{n-1} u(t) \, dt
\end{bmatrix}
\begin{bmatrix}
\int_0^T \alpha_0 u(t) \, dt \\
\cdot \\
\cdot \\
\int_0^T \alpha_{n-1} u(t) \, dt
\end{bmatrix}
\]

The above system has a solution if and only if the Kalman criterion is satisfied.
**Theorem**: A necessary and sufficient condition for the system (4) to be controllable is to have:

\[
\text{rank}(\delta) = \text{rank}([B|BA|BA^2|...BA^{n-1}]_{n \times nm}) = n
\]

**Proof of the Theorem will follow from Lemma 1 and 2.**

By solving the system (4) and evaluate at \(T\) we get:

\[
X(T) = e^{AT}x_0 + \int_0^T e^{A(T-t)}Bu_1(t)dt \quad (8)
\]

Now we find what \(u_1(t)\) that makes the controllability guaranteed. Without loosing generality, let \(x_0 = 0\) and therefore we use the transformation \(Y(T) = X(T) - e^{AT}x_0\).
Continuation of the proof

**Lemma 1**: In order for \( u_1(t) \) in Eq (8) to exist, we have to have the matrix \( g = \int_0^T C(t)C'(t)dt \) invertible, where \( C(t) = e^{A(T-t)}B \) and \( C'(t) \) is the transpose of \( C(t) \).

**Proof of Lemma 1**: Let \( u_1(t) = B'e^{A'(T-t)}g^{-1}Y(T) \) and by substituting in Eq (8) we get:

\[
\int_0^T e^{A(T-t)}Bu_1(t)dt = \int_0^T e^{A(T-t)}BB'e^{A'(T-t)}g^{-1}Y(T)dt
= gg^{-1}Y(T) = Y(T)
\] (9)
**Lemma 2**: Invertibility of $g$ is equivalent to have $\text{rank}(\delta) = n$.

**Proof of Lemma 2**: This proof is by contradiction.

Suppose that $g$ is not invertible, then there exists some $v \in \mathbb{R}^n$ with $v \neq 0$, such that $v'g = 0$. This implies that:

$$v'gv = \int_0^T v' C(t) C'(t) v dt = \int_0^T \left( C(t)v' \right)' \left( C(t)v' \right) dt = 0 \quad (10)$$

This implies that $v'C(t) = v'e^{A(T-t)}B = v' \left( I + \sum_{i \geq 1} A^i \frac{(T-t)^i}{i!} \right) B = 0$

We can see that since the above relation true for all $i$, then $v'\delta = 0$ with $v' \neq 0 \implies \text{rank}(\delta) \text{ strictly less than } n.$
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Consider the nonlinear system:

\[
\frac{dX}{dt} = f(X(t), u)
\]  

(11)

Then we linearize the system (11) about the equilibrium \((x_0, u_0)\) to be:

\[
\frac{dX}{dt} = AX + Bu
\]

where \(A = \frac{\partial f}{\partial x}(x_0, u_0)\) and \(B = \frac{\partial f}{\partial u}(x_0, u_0)\).

The system (11) is said to be first order controllable if \(A\) and \(B\) satisfy Kalman criterion.

**Remark**: First order (linearization) controllability implies local controllability BUT the converse is not true.
Let $M$ be a $C^\infty$ of dimension $m$ and $n \leq m$.

- For every $x \in M$, we assign an $n$-dimensional subspace $\Delta_x \subset T_x M$.
- There exists a neighborhood of $x$ in $M$ ($N_x \subset M$) such that there exist $n$ linearly independently vector fields $X_1, ..., X_n$ such that for every $y \in N_x$, we have $\Delta_y = \text{span}\{X_1, ..., X_n\}$.
- We call the collection of all $\Delta_x$ for all $x \in M$ a distribution $\Delta$.
- We say that a distribution $\Delta$ is involutive if for every point $x \in M$ there exists a local basis $\{X_1, ..., X_n\}$ in $N_x$ such that for all $1 \leq i, j \leq n$, we have the lie bracket $[X_i, X_j]$ is a linear combination of $\{X_1, ..., X_n\}$ ($[\Delta, \Delta] \subset \Delta$).
Local Controllability and Lie Brackets: Theorems and Mechanisms

**Theorem (Frobenius):** Suppose a distribution $\Delta$ has constant dimension. Then $\Delta$ is integrable if and only if $\Delta$ is involutive.

- Integrability can be helpful for reachability, **BUT** it rules out controllability.

Let us consider the drift-less system:

$$\frac{dX}{dt} = \sum_{i=1}^{m} g_i(X)u_i \quad (12)$$

- Let control distribution be $\Delta(x) = \text{span}\{g_1(x), \ldots, g_m(x)\}$.
- The above distribution on $M$ is said to be bracket generating if the iterated lie brackets $g_i, [g_i, g_j], [g_i, [g_j, g_k]], \ldots$ with $1 \leq i, j, k \leq m$ span the tangent space on $M$ at every point.
- The above distribution is bracket generating if and only if $\text{Lie}^\infty \{F\}_x = T_xM$ for every $x \in M$. 
Theorem (Rashevsky — Chow): Suppose that $M$ is connected. If the control distribution $\Delta(x) = \text{span}\{g_1(x), \ldots, g_m(x)\}$ is bracket generating, then the drift-less system in Eq (12) is controllable.

Remark: The Lie Algebra Rank Condition ($\text{Lie}^\infty \{F\}|_{x_0} = T_{x_0} M$) tells about accessibility from $x_0$ for systems with drift NOT controllability.

Remember:

Figure: Small Local time accessibility from $x_0$ if $R_\Sigma(x_0)$ has non empty interior.

Figure: Small Local time controllability from $x_0$ if $R_\Sigma(x_0)$ contains a neighborhood of $x_0$. 
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   - Theory and Applications
For drift-less systems, LARC \((\text{Lie}^\infty \{ F \}|_{x_0} = T_{x_0} M)\) is enough for STLC. For a system with a drift, another condition is required in addition to LARC to obtain STLC at \(x_0\).

**Definitions:**

- For a bracket \(B\), let \(|B|_0\) and \(|B|_a\), \(a \in \{1, \ldots, m\}\) denote the number of times the vector fields \(f\) and \(g_a\) appear in \(B\). For example \([(f, g_2), [g_1, [g_2, [f, g_2]]]]\) has \(|B|_0 = 2\), \(|B|_1 = 1\), \(|B|_2 = 3\), and \(|B|_i = 0\) for all \(i\) greater than 2.

- Let \(\beta\) be the set of potential obstruction to STLC. \(\beta = \{ B : |B|_0 \text{ is odd and } |B|_a \text{ is even } \forall a \in \{1, \ldots, m\}\} \).

- Let \(w = (w_0, \ldots, w_m)\) be a weight for each bracket such that \(||B||_w = \sum_{i=0}^{m} w_i |B|_i.\)

- A \(w\)-homogeneous element \(L \in \text{Lie}^\infty (F)\) is a linear combination of brackets having the same weight. For example \(w = (1, 1, 3)\), then \(||[g_1, [f, g_1]]||_w = 3\) and \(||g_2||_w = 3\) as well, as a result, \(L = g_2 + [g_1, [f, g_1]]\) is \(w\)-homogeneous.
Let $S_m$ be the set of permutations on $m$-symbols, $\sigma \in S_m$ acts only on the last $m$ components of $w$, by interchanging the weights $(\sigma(w_0, w_1, ..., w_m) = (w_0, w_{\sigma(1)}, ..., w_{\sigma(m)}))$. We have $\sigma \in S_m^w$ is $\sigma(w) = w$. For example $m = 2$ and $w = (0, 1, 1)$ have $\sigma(1, 2) = (2, 1) \in S_m^w$ unlike the case if $w = (0, 1, 3)$.

Define the set $\beta^w_s = \{ L \in \beta : \sigma(L) = L \ \forall \sigma \in S_m^w \}$ such that $w$-obstructions is defined as $\beta^w = \text{Lie}^\infty(f, \beta^w_s)$. For example: if $m = 2$, $w = (0, 1, 1)$, then $L = [g_1, [f, g_1]] + [g_2, [f, g_2]] \in \beta^w_s ([g_1, [f, g_1]], [g_2, [f, g_2]] \not\in \beta^w_s)$.

**Theorem**: The control-affine system $\Sigma = (m, F, U)$ is STLC from $x_0 \in M$ if:

- $\Sigma$ satisfies the LARC at $x_0$.
- There exists admissible weight $w$ such that each $w$-homogeneous element of $\beta^w$ is neutralized at $x_0$. 
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STLC in literature

A summary of some of the work done on STLC.
Sufficient condition:

- M. I. Krastanov 2009.

Necessary condition:


Necessary and Sufficient conditions for some specific classes of systems:

- C. O. Aguilar and A. D. Lewis 2012.
Problems in the theory:

- No necessary and sufficient conditions.
- No degree of controllability definitions in the case of nonlinear systems.
- No rigorous work to relate controllability and stability, especially for nonlinear systems.
- Advances of GCT require advances with computational tools as well.

Applications:

- Flight dynamics.
- Fluid dynamics.
- Renewable energies control systems.
Thanks!

Thank you for listening! Will be glad to answer any questions.